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Theories of the Action of Magnetism on Light.

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1. About five years ago I worked out a theory of the reflection and refraction of light at the surface of a magnetized *transparent* medium* by means of a suggestion of Professor Rowland, which consisted in introducing Hall's effect into the general equations of the electro-magnetic field. I did not, however, at the time attempt to apply the results to the phenomenon of reflection from a magnetized *metal*, for being aware that metallic reflection could not be explained upon the electro-magnetic theory by taking into consideration the conductivity of the metal, and that no satisfactory theories of metallic reflection existed

*Phil. Trans. 1891, A, p. 371; Physical Optics, Chap. XX.

except those in which the ether is regarded as an elastic medium, I was uncertain how to proceed. I subsequently determined to try what could be done by transforming the expressions for the intensities of the reflected waves in the same manner as Cauchy and Eisenlohr transformed Fresnel's sine and tangent formulæ;* and I found that the agreement between the theoretical results thus obtained and the experiments of Kerr† was perfect as far as qualitative results were concerned, and in one case the theory showed the existence of a phenomenon which Kerr had failed to detect, but which was shortly afterwards discovered by Kundt‡ upon repeating Kerr's experiments. The agreement as regards quantitative results was not, however, in all cases, so close as might be desired.

But irrespective of the question of metallic reflection, the theory was not entirely satisfactory, since it required the tangential component of the electro-motive force to be discontinuous at the surface of separation of a magnetized and an unmagnetized medium. I was fully aware of this objection at the time, and endeavored to explain the difficulty by suggesting that possibly the transition from one medium to the other was not abrupt, but that a thin interfacial layer might exist through which there was a rapid but continuous change in the value of the tangential component of the electro-motive force. I must however confess that this explanation savors of a device for evading rather than accounting for a difficulty.

2. Mr. Larmor|| has recently attempted to resuscitate a modification of Maxwell's theory,§ which was proposed in 1879 by Professor Fitz Gerald.¶ Fitz Gerald's theory contains a serious defect owing to the boundary conditions being too numerous; and I accordingly awaited the publication of Mr. Larmor's papers with much interest, in the hope that he had succeeded in overcoming the difficulties to which I have alluded. But in this I was disappointed, for the papers in question, instead of containing a careful mathematical investigation of a phenomenon whose interest is equal to the difficulty of satisfactorily

* *Proc. Camb. Phil. Soc.*, vol. VIII, p. 68.

† *Phil. Mag.*, May 1877; *Ibid.* March 1878.

‡ *Phil. Mag.*, Oct. 1884; and see *Physical Optics*, Ch. XX.

|| *Brit. Assoc. Rep.* 1893, p. 335; *Phil. Trans.* 1894, A, Pt. II, p. 779.

§ *Electricity and Magnetism*, Chap. XXI.

¶ *Phil. Trans.* 1880, p. 691.

accounting for it, consisted for the most part of vague and obscure generalities, which are calculated to envelop the subject in a cloud of mystery rather than to enlighten the understanding. The superficial reader may possibly be impressed with their apparent profundity, but when examined they turn out to be a dry husk without a kernel.

3. The object of the present communication is two-fold. In the first place I shall subject Mr. Larmor's theory to a searching examination for the purpose of exposing its imperfections, and shall show that instead of being an improvement on its predecessors it is open to a variety of additional objections and defects. In the second place I shall show that by means of a slight modification of the fundamental hypothesis, the theory of Rowland and myself may be placed on a perfectly satisfactory basis, and that the difficulty with regard to the discontinuity of the tangential component of the electro-motive force at an interface may be removed.

The Theory of Fitz Gerald and Larmor.

4. In Maxwell's general theory of the electro-magnetic field, the electric displacement and the magnetic force are connected by the equations

$$4\pi\dot{f} = \frac{d\gamma}{dy} - \frac{d\beta}{dz}, \quad 4\pi\dot{g} = \frac{d\alpha}{dz} - \frac{d\gamma}{dx}, \quad 4\pi\dot{h} = \frac{d\beta}{dx} - \frac{d\alpha}{dy}, \quad (1)$$

whilst the electro-kinetic energy T and the electro-static energy W per unit of volume are given by the equations

$$T = \frac{\mu}{8\pi} (\alpha^2 + \beta^2 + \gamma^2), \quad (2)$$

$$W = \frac{2\pi}{K} (f^2 + g^2 + h^2). \quad (3)$$

Fitz Gerald introduces a new vector $\mathfrak{B}$, whose time variation is the magnetic force, so that if ξ, η, ζ be its components,

$$\dot{\xi} = \alpha, \quad \dot{\eta} = \beta, \quad \dot{\zeta} = \gamma, \quad (4)$$

whence it follows from (1) that

$$4\pi\dot{f} = \frac{d\zeta}{dy} - \frac{d\eta}{dz}, \quad 4\pi\dot{g} = \frac{d\xi}{dz} - \frac{d\zeta}{dx}, \quad 4\pi\dot{h} = \frac{d\eta}{dx} - \frac{d\xi}{dy}, \quad (5)$$

so that the electric displacement is proportional to the curl of $\mathfrak{B}$.

The electro-kinetic energy is therefore proportional to the square of the time variation of $\mathfrak{B}$, whilst the electro-static energy depends upon its space variations. Under these circumstances the quantity $T - W$ is a function of ξ, η, ζ and their differential coefficients, and Fitz Gerald proceeds to apply the Calculus of Variations and the Principle of Least Action to obtain the equations of motion and the boundary conditions.

To account for magnetic action upon light, Fitz Gerald, following Maxwell, assumes that the kinetic energy of the medium contains an additional term depending upon the displacement of certain supposed vortices, which is of the form

$$T' = 4\pi C \left(\frac{d\xi}{d\theta} \dot{f} + \frac{d\eta}{d\theta} \dot{g} + \frac{d\zeta}{d\theta} \dot{h} \right), \quad (6)$$

where

$$\frac{d}{d\theta} = \alpha_0 \frac{d}{dx} + \beta_0 \frac{d}{dy} + \gamma_0 \frac{d}{dz} \quad (7)$$

and $\alpha_0, \beta_0, \gamma_0$ denote the components of the external magnetic force.

Fitz Gerald found that the introduction of this additional term into the kinetic energy created a difficulty with regard to the boundary conditions, which Larmor proposes to deal with in the following manner. The components ξ, η, ζ of the vector $\mathfrak{B}$ satisfy the solenoidal condition $d\xi/dx + d\eta/dy + d\zeta/dz = 0$; consequently their variations are not independent but are subject to the above condition. Larmor therefore introduces into the equation of Least Action the additional term

$$\lambda \left(\frac{d\xi}{dx} + \frac{d\eta}{dy} + \frac{d\zeta}{dz} \right),$$

where λ is an undetermined function of x, y, z . He thus obtains the following equations of motion:

$$\begin{aligned} \mu \frac{d^2 \xi}{dt^2} = \frac{1}{K} \left\{ \frac{d}{dz} \left(\frac{d\xi}{dz} - \frac{d\zeta}{dx} \right) - \frac{d}{dy} \left(\frac{d\eta}{dx} - \frac{d\xi}{dy} \right) \right\} \\ - 8\pi C \frac{d}{d\theta} \left(\frac{d\zeta}{dy} - \frac{d\eta}{dz} \right) - 4\pi \frac{d\lambda}{dx} \quad (8) \end{aligned}$$

with two similar ones. From these equations it follows that $\nabla^2 \lambda = 0$.

5. Larmor has not attempted to examine the physical meaning of these equations or to enquire how far they agree with those furnished by other theories; and this is what I shall now do.

From (4) and (5) it follows that (8) may be expressed in the form

$$\frac{da}{dt} = \frac{4\pi}{K} \left(\frac{dg}{dz} - \frac{dh}{dy} \right) - 32\pi^2 C \frac{df}{d\theta} - 4\pi \frac{d\lambda}{dx}. \quad (9)$$

With the exception of the additional term $d\lambda/dx$, the above equation, which determines the time variation of the magnetic induction, is of the same form as equation (10) of my paper in the Phil. Trans. 1891, as can be seen by recollecting that for an isotropic medium $B^2 = C^2 = (\mu K)^{-1}$.

If we eliminate λ we shall obtain three equations of the form

$$\frac{d^2 f}{dt^2} = \frac{1}{K\mu} \nabla^2 f + 8\pi C \frac{d}{d\theta} \left(\frac{dg}{dz} - \frac{dh}{dy} \right). \quad (10)$$

These are the equations which determine the electric displacement, and they are of the same form as equations (12) of my paper referred to above, from which it follows that, *so far as the propagation of light is concerned*, both theories lead to the same results. The value of the magnetic force is, however, not the same unless $\lambda = 0$; and Larmor has shown that λ cannot be zero, but is the potential of a surface distribution upon interfaces.

6. There is now a further point to be considered. The introduction of an additional term into the kinetic energy must necessarily produce an alteration in some of Maxwell's general equations of the electro-magnetic field. We must therefore enquire, which of the various sets of equations are modified? What are the additional terms introduced? Can any experimental evidence be brought forward in support of the existence of the additional terms whose presence is a necessary consequence of the theory? But upon all these interesting and important questions Mr. Larmor preserves an impenetrable silence, and not a single hint is given with regard to their solution.

To examine these questions, we must first recall to mind Maxwell's general equations of the electro-magnetic field. They consist of the following four sets:

$$P = -\frac{dF}{dt} - \frac{d\psi}{dx}, \quad (11)$$

$$P = 4\pi f/K, \quad (12)$$

$$a = \frac{dH}{dy} - \frac{dG}{dz}, \quad (13)$$

$$4\pi\mu\dot{f} = \frac{dc}{dy} - \frac{db}{dz}, \quad (14)$$

where μ is the magnetic permeability.

Equation (14) cannot be altered, since it is assumed in the theory. Equation (13) is a mere analytical artifice which amounts to introducing a quantity, called by Maxwell the vector potential, whose curl is the magnetic induction. The vector potential is essentially an indeterminate quantity, for if the magnetic induction is given, and F , G , H be any particular values which satisfy (13), the complete values are $F + d\phi/dx$, etc., where ϕ is some function of (x, y, z, t) . The indeterminateness of the vector potential is taken account of in (11) by the introduction of the quantity ψ , and the ambiguity involved in its employment may be got rid of by its elimination.

Equation (12) is implicitly assumed in the expression for the electro-static energy of the medium, which is supposed to be of the form given by Maxwell. The additional term introduced into the energy for the purpose of explaining electro-optical effects is kinetic and not static.

7. Under these circumstances equation (11) is the only one capable of modification. We shall therefore assume that

$$P = -\frac{dF}{dt} - p_3\dot{g} + p_2\dot{h} - \frac{d\psi}{dx} + L, \quad (15)$$

etc., where $p_1 = 32\pi^2 C\alpha_0$, etc. This hypothesis is equivalent to the introduction of Hall's effect into the general equations of E. M. F., together with additional terms L , M , N .

From (13) and (15) we obtain

$$\frac{da}{dt} = \frac{4\pi}{K} \left(\frac{dg}{dz} - \frac{dh}{dy} \right) - 32\pi^2 C \frac{df}{d\theta} + \frac{dN}{dy} - \frac{dM}{dz}. \quad (16)$$

Comparing this with (9) it follows that

$$-4\pi \frac{d\lambda}{dx} = \frac{dN}{dy} - \frac{dM}{dz}. \quad (17)$$

Let Φ be a solution of Laplace's equation, and let

$$-4\pi\lambda = \Phi + x \frac{d\Phi}{dx} + y \frac{d\Phi}{dy} + z \frac{d\Phi}{dz}, \quad (18)$$

$$\left. \begin{aligned} L &= z \frac{d\Phi}{dy} - y \frac{d\Phi}{dz}, \\ M &= x \frac{d\Phi}{dz} - z \frac{d\Phi}{dx}, \\ N &= y \frac{d\Phi}{dx} - x \frac{d\Phi}{dy}, \end{aligned} \right\} \quad (19)$$

then

$$\frac{dN}{dy} - \frac{dM}{dz} = \frac{d}{dx} \left(\Phi + x \frac{d\Phi}{dx} + y \frac{d\Phi}{dy} + z \frac{d\Phi}{dz} \right) \quad (20)$$

$$= -4\pi \frac{d\lambda}{dx}. \quad (21)$$

Since the right-hand side of (18) is a solution of Laplace's equation, $\nabla^2\lambda = 0$ as ought to be the case; whence the requirements of the theory are satisfied by introducing Hall's effect together with the additional terms L , M , N whose values are given by (19). There does not, however, appear to be any evidence, experimental or otherwise, which would justify the introduction of these additional terms into the general equations of E. M. F., and unless this question can be satisfactorily answered, the theory is open to objections from which its predecessors are free.

8. In order to examine the boundary conditions, Larmor takes the axis of z perpendicular to the surface of separation which is supposed to be plane, and he obtains a result which by virtue of (4) and (5) is equivalent to the condition that

$$\begin{aligned} \left(\frac{4\pi g}{K} + 4\pi C \frac{d\beta}{d\theta} - 16\pi^2 C \gamma_0 \frac{df}{dt} \right) \delta\xi - \left(\frac{4\pi f}{K} + 4\pi C \frac{d\alpha}{d\theta} + 16\pi^2 C \gamma_0 \frac{dg}{dt} \right) \delta\eta \\ - \left(\lambda + 16\pi^2 C \gamma_0 \frac{dh}{dt} \right) \delta\zeta \end{aligned} \quad (22)$$

should be continuous.

When the direction of propagation of a plane wave of light lies in the plane (xz), the continuity of the normal component of electric displacement necessitates the continuity of β ; we are therefore forced for the sake of consistency to suppose that α is continuous. Accordingly both the tangential components of the magnetic force must be continuous, which requires that $\delta\xi$ and $\delta\eta$ should be con-

tinuous, and it therefore follows that the coefficients of these quantities must be continuous.

Now, according to Maxwell's theory of the electro-magnetic field,

$$P = 4\pi f/K, \quad Q = 4\pi g/K,$$

whence recollecting that in an unmagnetized medium $C = 0$, and denoting the values of quantities in the latter medium by accented letters, the continuity of the coefficients of $\delta\xi$, $\delta\eta$ lead to the two boundary conditions

$$\left. \begin{aligned} Q + 4\pi C \left(\frac{d\beta}{d\theta} - 4\pi\gamma_0 \frac{df}{dt} \right) &= Q', \\ P + 4\pi C \left(\frac{d\alpha}{d\theta} + 4\pi\gamma_0 \frac{dg}{dt} \right) &= P', \end{aligned} \right\} \quad (23)$$

in other words, *the tangential components of the electro-motive force are discontinuous.*

With regard to the last boundary condition, it follows from Maxwell's theory that the normal component of the magnetic induction must be continuous; whence $\mu\delta\zeta = \mu'\delta\zeta'$, so that we obtain from (22)

$$\mu' \left(\lambda + 16\pi^2 C\gamma_0 \frac{df}{dt} \right) = \mu\lambda'.$$

In an unmagnetized medium $\lambda' = 0$, whence at the surface of separation

$$\lambda = -16\pi^2 C\gamma_0 df/dt, \quad (24)$$

so that λ is the potential of a surface distribution of electricity in the interface whose surface value is determined by (24).

9. Larmor, in summing up the results of his theory on p. 349 of the British Association Report for 1893, says: "It appears therefore that we have here a consistent scheme of equations of reflection and refraction without the necessity of condoning any dynamical difficulties in the process, the result being in all respects implicitly involved in the expression for the energy function of the medium." And again, on p. 359, he says: "But against this procedure" (i. e. my own theory) "there stands the pure assumption as regards the discontinuity of electric force at an interface. The correct boundary conditions would be derived from the modification of Fitz Gerald's procedure, which has been explained above." And in an exceedingly verbose paper in the Philosophical

Transactions for 1894 he returns to the subject (p. 780) and says: "I have elsewhere tried to show that, on a consensus of various reasons, this term, originally given by Maxwell, must be taken as the correct representation of the magneto-optic effect." But the preceding analysis shows that there is no justification for the favorable view with which this theory is regarded by its advocate, for when it is stripped of the verbiage in which it is enveloped and subjected to a searching examination, the theory is not only found to be open to the same objections as the one proposed by Rowland and myself—viz. discontinuity of the tangential component of the E. M. F. at an interface—but possesses the further defect of requiring the introduction of certain additional terms into the general equations of E. M. F. for which there is no apparent justification whatever.

We may further enquire, what is the physical meaning of equations (23)? But to this question Mr. Larmor vouchsafes no reply. The equations are a mere aggregation of symbols, to which no meaning other than an analytical one can apparently be attached.

Modification of my Former Theory.

10. The theory of magnetic action which was developed in my paper in the Phil. Trans. may be placed on a satisfactory basis by a modification of the fundamental hypothesis.

According to Maxwell's general theory of the electro-magnetic field, the two equations which involve the electro-motive force are

$$F = - \dot{F} - d\psi/dx, \quad (1)$$

$$P = 4\pi f/K, \quad (2)$$

the first of which is the analytical expression for the theorem that the electro-motive force round a closed circuit is equal to the rate of decrease of the number of lines of magnetic induction passing through the circuit, whilst the second asserts that when electro-motive force acts upon a dielectric it produces an electric displacement which is proportional to the E. M. F. Now the hypothesis adopted by Rowland was that when light is propagated through a transparent medium which is under the influence of a magnetic field, the latter produces an effect analogous to Hall's effect, which may be represented mathematically by introducing the additional terms $-p_3\dot{g} + p_2\dot{h}$, etc., into equation (1). We may, however, in the absence of any experimental evidence to the contrary, assume

that the effect of the magnetic field may be equally well represented by introducing the term $p_3g - p_2\dot{h}$ into equation (2), and if this be done it will be found that the difficulty with regard to the discontinuity of the E. M. F. at an interface is removed, and that all the equations furnish a consistent scheme in which none of the fundamental principles of dynamics are violated.

11. For the sake of generality we shall suppose that the medium is a doubly refracting one, in which case the relations between electro-motive force and electric displacement will be given by the equations

$$\left. \begin{aligned} P &= 4\pi f / K_1 + p_3\dot{g} - p_2\dot{h}, \\ Q &= 4\pi g / K_2 + p_1\dot{h} - p_3\dot{f}, \\ R &= 4\pi h / K_3 + p_2\dot{f} - p_1\dot{g}, \end{aligned} \right\} \quad (3)$$

where $p_1 = C\alpha_0$, etc., $\alpha_0, \beta_0, \gamma_0$ being the components of the external magnetic force and C a constant.

All the other equations are the same as Maxwell's (with the exception that we do not suppose that F, G, H satisfies the solenoidal condition), and are therefore

$$P = -\dot{F} - d\psi/dx, \text{ etc.}, \quad (4)$$

$$a = \frac{dH}{dy} - \frac{dG}{dz}, \text{ etc.}, \quad (5)$$

$$4\pi\mu f = \frac{dc}{dy} - \frac{db}{dz}, \text{ etc.}, \quad (6)$$

where μ is the magnetic permeability.

$$\text{Let} \quad \mu K_1 = A^{-2}, \quad \mu K_2 = B^{-2}, \quad \mu K_3 = C^{-2}, \quad (7)$$

$$\Omega = A^2 \frac{df}{dx} + B^2 \frac{dg}{dy} + C^2 \frac{dh}{dz}, \quad (8)$$

$$\frac{d}{d\omega} = p_1 \frac{d}{dx} + p_2 \frac{d}{dy} + p_3 \frac{d}{dz}. \quad (9)$$

From (4) and (5) we obtain

$$\frac{da}{dt} = \frac{dQ}{dz} - \frac{dR}{dy}.$$

Substituting the values of Q and R from (3), we obtain

$$\frac{da}{dt} = 4\pi\mu \left(B^2 \frac{dg}{dz} - C^2 \frac{dh}{dz} \right) - \frac{d\dot{f}}{d\omega}. \quad (10)$$

This is the equation which connects the magnetic induction with the electric displacement, and it is identical with equation (10) of my paper in the Phil. Trans.

Differentiating (6) with respect to the time, and substituting the values of da/dt , etc., from (10), we obtain three equations of the form

$$\frac{d^2 f}{dt^2} = A^2 \nabla^2 f - \frac{d\Omega}{dx} + \frac{1}{4\pi\mu} \frac{d}{d\omega} \left(\frac{d\dot{g}}{dz} - \frac{d\dot{h}}{dy} \right), \quad (11)$$

which are exactly the same as equations (12) in the Phil. Trans.

So far as the propagation of light is concerned, the modified theory leads to exactly the same results as the original one.

The Boundary Conditions.

12. Let the plane $x = 0$ be the surface of separation, and let the plane $z = 0$ contain the direction of propagation. Then, as shown in my paper in the Phil. Trans., the continuity of f , the normal component of the electric displacement, involves the continuity of γ , which is one of the tangential components of the magnetic force.

The continuity of a , the normal component of the magnetic induction, involves the continuity of H ; for since none of the quantities are functions of z , $a = dH/dy$ from (5). For the same reason it follows from (4) that R , which is one of the tangential components of the electro-motive force, must also be continuous, and from the last of (3) this leads to the condition

$$4\pi h/K_3 + p_2 \dot{f} - p_1 \dot{g} = 4\pi h'/K', \quad (12)$$

where the accented letters refer to the unmagnetized medium.

The mathematical form of this equation is exactly the same as equation (17) of my paper in the Phil. Trans.; accordingly the modified hypothesis shows that the continuity of the normal component of the magnetic induction leads to the continuity of *one* of the tangential components of the E. M. F.

The two remaining boundary conditions must necessarily be continuity of the other tangential components of magnetic force and electro-motive force. The

latter condition is represented by the equation

$$4\pi g/K_2 + p_1\dot{h} - p_3\dot{f} = 4\pi g'/K', \quad (13)$$

which is the same as (19) of my paper in the *Phil. Trans.*

We therefore see that the above modification of the original theory leads to a thoroughly consistent scheme of equations, which furnish equivalent results both as regards the representation of the phenomena in the interior of the medium and also as regards the boundary conditions. At the same time the modified theory is free from the defect of violating Newton's third Law of Motion, since the tangential components of the electro-motive force and the magnetic force are both continuous at an interface.

13. The only remaining point to be considered is the deduction of the foregoing results by means of the Principle of Least Action. Mr. Larmor has argued quite correctly that if the value of the energy function is known, the equations of motion and the boundary conditions must necessarily be capable of being deduced therefrom by the Principle of Least Action; but the fallacy of his argument—a fallacy which pervades the whole of his paper in the *Philosophical Transactions* for 1894, as well as his Report to the British Association in 1893—is that he seems to imagine that if some form of the energy function is postulated, which is capable of furnishing the correct equations of motion and also a sufficient number of equations for determining all the unknown quantities, this function must necessarily be the correct one. The fallacy of this line of reasoning is at once apparent from the fact that if some particular form of the energy function is postulated which furnishes sufficient equations to determine all the unknown quantities, but at the same time leads to results which violate any of the fundamental principles of dynamics, the theory is dynamically unsound, and the results represent some impossible form of motion. To write down certain mathematical expressions and to perform certain mathematical operations do not constitute a satisfactory theory, unless it can be shown that all the results lead to a consistent scheme in which none of the fundamental principles of dynamics are violated.

Let T and U denote Maxwell's expressions for the electro-kinetic and electro-static energies, so that

$$T = \frac{\mu}{8\pi} (\alpha^2 + \beta^2 + \gamma^2), \quad (14)$$

$$U = 2\pi\mu (A^2f^2 + B^2g^2 + C^2h^2), \quad (15)$$

also let

$$U' = \frac{1}{2} \{ (p_3 \dot{g} - p_2 \dot{h}) \dot{f} + (p_1 \dot{h} - p_3 \dot{f}) \dot{g} + (p_2 \dot{f} - p_1 \dot{g}) \dot{h} \}, \quad (16)$$

and consider the variation of the energy function

$$\delta \int dt \int (T - U - U') d\tau = 0, \quad (17)$$

where $d\tau$ denotes an element of volume.

We have at once

$$\begin{aligned} \delta \int dt \int U' d\tau &= \frac{1}{2} \int dt \int \{ (p_3 \dot{g} - p_2 \dot{h}) \delta \dot{f} - (p_3 \dot{g} - p_2 \dot{h}) \delta \dot{f} + \dots \} d\tau \\ &= \int dt \int \{ (p_3 \dot{g} - p_2 \dot{h}) \delta \dot{f} + (p_1 \dot{h} - p_3 \dot{f}) \delta \dot{g} + (p_2 \dot{f} - p_1 \dot{g}) \delta \dot{h} \} d\tau \\ &\quad + \text{an integral depending on the limits of the time.} \end{aligned}$$

Introducing the quantities ξ, η, ζ , where the time variations of these quantities are the magnetic force, and recollecting that from (6)

$$4\pi f = \frac{d\zeta}{dy} - \frac{d\eta}{dz}, \text{ etc.}, \quad (18)$$

we obtain

$$\begin{aligned} \delta \int dt \int U' d\tau &= \frac{1}{4\pi} \int dt \int \left\{ (p_3 \dot{g} - p_2 \dot{h}) \left(\frac{d\delta \zeta}{dy} - \frac{d\delta \eta}{dz} \right) + \dots \right\} d\tau \\ &= \frac{1}{4\pi} \int dt \int [\{ n (p_1 \dot{h} - p_3 \dot{f}) - m (p_2 \dot{f} - p_1 \dot{g}) \} \delta \xi + \dots] dS \\ &\quad + \frac{1}{4\pi} \int dt \int \left\{ \frac{d\dot{f}}{d\omega} \delta \xi + \frac{d\dot{g}}{d\omega} \delta \eta + \frac{d\dot{h}}{d\omega} \delta \zeta \right\} d\tau, \end{aligned} \quad (19)$$

where dS is an element of the boundary and l, m, n its direction cosines.

Working out the variation of the term $T - U$ in the same way, we shall find that the equations which hold good in the interior of the medium are

$$\frac{da}{dt} = 4\pi\mu \left(B^2 \frac{dg}{dz} - C^2 \frac{dh}{dy} \right) - \frac{d\dot{f}}{d\omega}, \quad (20)$$

whilst the surface integral term is

$$\int dt \int [\{ 4\pi\mu (nB^2g - mC^2h) + n (p_1 \dot{h} - p_3 \dot{f}) - m (p_2 \dot{f} - p_1 \dot{g}) \} \delta \xi + \dots] dS,$$

which, by virtue of (3) and (7), may be written

$$\int dt \int [(nQ - mR) \delta\xi + (lR - nP) \delta\eta + (mP - lQ) \delta\zeta] dS. \quad (21)$$

Equation (20) is identical with (10), so that the proposed energy function leads to the equations which must hold good in the interior of the medium. With regard to the surface integral, we shall suppose as before that the plane $x = 0$ is the surface of separation; accordingly $m = n = 0$, $l = 1$, and the boundary conditions require that the expression

$$R\delta\eta - Q\delta\zeta$$

should be continuous. The continuity of the tangential component of the magnetic force at an interface requires that $\delta\eta$ and $\delta\zeta$ should be continuous, and this leads to the condition that R and Q must be continuous. In other words, the tangential components of the electro-motive force must be continuous.

It is unnecessary to introduce the term $\lambda (d\xi/dx + d\eta/dy + d\zeta/dz)$ into the equation of Least Action, since the foregoing analysis shows that $\lambda = 0$.

14. We must now consider the meaning of the additional term U' .

From the particular manner in which this term has been introduced, it follows that the energy, which is supposed to be due to magnetic action, is electro-static and not electro-kinetic, and from (3) it follows that

$$U + U' = \frac{1}{2}(Pf + Qg + Rh),$$

so that the electro-static energy produced by electro-motive force may still be regarded as equal to half the product of the force into the electric displacement; but the E. M. F. contains certain additional terms which depend upon the superimposed magnetic force. There seems to be some ground for supposing that the additional term in the energy ought to be statical rather than kinetic, owing to the fact that the motion consists of oscillations of about a state of strain produced by magnetic force, and this would necessarily lead to a statical term in the potential energy which arises from the statical effects of magnetic force.

15. If in the expression for U' we substitute the values of $\dot{f}$, $\dot{g}$, $\dot{h}$ in terms of the magnetic force, we obtain

$$U' = \frac{1}{8\pi} \left\{ f \frac{d\alpha}{d\omega} + g \frac{d\beta}{d\omega} + h \frac{d\gamma}{d\omega} - \left(f \frac{d}{dx} + g \frac{d}{dy} + h \frac{d}{dz} \right) (p_1\alpha + p_2\beta + p_3\gamma) \right\}.$$

Integrating the last term by parts, it will be found that the volume integral vanishes and we get

$$\int U' d\tau = \frac{1}{8\pi} \int \left(f \frac{d\alpha}{d\omega} + g \frac{d\beta}{d\omega} + h \frac{d\gamma}{d\omega} \right) d\tau \\ - \int (lf + mg + nh)(p_1\alpha + p_2\beta + p_3\gamma) dS.$$

The volume integral part is of the same form as one of the expressions suggested by Mr. Larmor,* and it is worthy of note that by integrating with respect to the time this expression can be converted into the one proposed by Fitz Gerald. So far as the internal equations are concerned, it is immaterial which of the three expressions we employ, but the preceding analysis shows that the expression for U' employed in the present paper is the only one yet proposed which leads to a "consistent scheme without the necessity of condoning any dynamical difficulties in the process."

FLEDBOROUGH HALL, HOLYPORT, BERKS, Nov. 30, 1895.

* Brit. Assoc. Rep. 1893, p. 347.